Discrete Mathematics 39 (1982) 229–234
North-Holland Publishing Company

NOTE

A CATALOGUE OF SMALL MAXIMAL NONHAMILTONIAN GRAPHS

Janusz JAMROZIK

Department of Genetics and Animal Breeding, Academy of Agriculture (AR), Mickiewicza 24, 30–059 Kraków, Poland

Rafal KALINOWSKI

Institute of Mathematics, Academy of Minining and Metallurgy (AGH), Mickiewicza 30, 30–059 Kraków, Poland

Zdzislaw SKUPIEŃ*

Mathematics Department, Kuwait University, P.O. Box 5969, Kuwait (on leave from AGH, Kraków)

Received 30 March 1979
Revised 15 August 1979 and 5 January 1981

In this paper a catalogue of all maximal nonhamiltonian graphs of orders up to 10 is provided. Special attention is paid to maximal nonhamiltonian graphs with non-positive scattering number since all remaining ones (with scattering number 1) are fully characterized and counted by the third author. We also give a sketch of the method used to produce the catalogue.

1. Preliminaries

Unless otherwise stated, we use standard notation and terminology of graph theory. All graphs we deal with are *simple*. Throughout, n stands for the *order* of a graph G (or of graphs we deal with). Only if G_1 and G_2 are *disjoint graphs*, we write $G_1 \cup G_2$ and $G_1 * G_2$ to denote their *union* and *join*, respectively. If $G \subseteq H$ and $V(G) = V(H)$ then G is a *factor* of H and H is a *counterfactor* of G. A block G is a *minimal block* if each of its proper factors either is disconnected ($n = 2$) or has a cut-vertex ($n \geq 3$).

A nonhamiltonian graph is called MNH (*maximal nonhamiltonian*) if it either is a complete graph K_1 or K_2 or becomes hamiltonian after the addition of any new edge. In other words, a nonhamiltonian graph G is maximal if and only if any two nonadjacent vertices are connected by a hamiltonian path.

* The third author was partially supported by the Kuwait University Research Council Grant, No. SM 003.

Following Jung [4], we define the *scattering number* $s(G)$ of G as

$$s(G) = \max\{k(G-S)-|S| : S \subseteq V(G), k(G-S) \neq 1\}$$

where $k(G-S)$ stands for the *number of components* of $G-S$. Note that $s(G) \leqslant 0$ whenever G is hamiltonian (or if and only if G is 1-*tough* [2]). We find the scattering number more convenient than the notion of toughness for describing MNH graphs.

2. Known basic results

The following known properties of maximal nonhamiltonian graphs will be useful in what follows.

Theorem (cf. [5]). *Let G be a MNH graph of order n. Then the following properties hold.*

Property 1. *G is connected and, for $n \geqslant 3$, the connectivity $\kappa(G)$ of G satisfies the inequality* $1 \leqslant \kappa(G) \leqslant \frac{1}{2}(n-1)$.

Property 2. *For any two vertices u, v of G, if* $\deg(u)+\deg(v) \geqslant n$ *then* $uv \in E(G)$.

Property 3. $\Delta(G) = n-1$ *or, only for* $n \geqslant 9$, $\Delta(G) \leqslant n-4$ and $s(G) \leqslant 0$.

Property 4. $s(G) \leqslant 1$ *where the equality holds true if and only if* $n \geqslant 3$ *and there is an integer* κ *with* $1 \leqslant \kappa \leqslant \frac{1}{2}(n-1)$ *such that there is a partition* $(n_i)_{i=1}^{\kappa+1}$ (*where* $n_j \geqslant n_{j-1}$ *for* $j = 2, 3, \ldots, \kappa+1$) *of* $n-\kappa$ *into* $\kappa+1$ *parts such that*

$$G = K_\kappa^{(0)} * \bigcup_{i=1}^{\kappa+1} K_{n_i}^{(i)} \tag{1}$$

where $K_\kappa^{(0)}, K_{n_1}^{(1)}, \ldots, K_{n_{\kappa+1}}^{(\kappa+1)}$ *form a set of* $\kappa+2$ *mutually disjoint complete graphs* ($K_{n_i}^{(i)}$ *denotes the ith complete graph of order* n_i).

Note that the upper bound for $\kappa(G)$ in Property 1 follows from the well-known Dirac's theorem of 1952 on the existence of hamiltonian circuits. Analogously, Property 2 is a simple consequence of the famous Ore's result of 1960 and reads in terminology of [1] that the n-closure of a MNH graph G is G itself. Furthermore, condition $\Delta(G) \leqslant n-4$ in Property 3 can be replaced by the stronger one: $\delta(G)+\Delta(G) \leqslant n-2$ and $\delta(G) \geqslant 2$ (cf. [6]).

3. 1-tough MNH graphs

Since Property 4 explicitly describes all non-1-tough MNH graphs, we restrict our attention to remaining MNH graphs of order $n \geqslant 3$ which are 2-connected. Hence each of them contains a nonhamiltonian minimal block. Since the list of minimal blocks of orders at most 10 is available in Hobbs [3], completing the corresponding list of MNH graphs can consist in finding all MNH counterfactors of each nonhamiltonian item of Hobbs' list. This idea together with Property 4 was used by Skupień [4] to produce the list of all MNH graphs with $n \leqslant 7$. Because the number, say b_n, of nonhamiltonian minimal blocks of order n increases rather rapidly with n (see Table 1, derived from [2]), we have extended Skupień's list with the help of a computer.

Table 1. Numbers of nonhamiltonian minimal blocks.

n	1	2	3	4	5	6	7	8	9	10
b_n	1	1	0	0	1	2	5	11	27	67

An essentially backtrack algorithm for finding the main upper triangles of the adjacency matrices of 1-tough MNH graphs G with $7 \leqslant n \leqslant 10$ has been used. In the computer algorithm at each stage of the process of augmentation, before trying to add a new edge to a given block, the block is replaced by its n-closure first. Some essential modifications are introduced to reduce the time of execution of the computer program. For instance, all MNH graphs G with $\Delta(G) = n-1$ $(n \geqslant 5)$ are generated from two special factors. Therefore blocks are being augmented only to graphs G with $\Delta(G) \leqslant n-4$. We omit further details.

4. The catalogue

In order to spare space we avoid much of picture drawing because the structure of many our 1-tough MNH graphs can easily be described. First we describe A-graphs. Namely, there exist $r \times s$ 0-1 matrices $A = [a_{ij}]$, the following three matrices A_α $(\alpha = 1, 2, 3)$ if $n \leqslant 10$ (with $r = s-1$):

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which together with ordered partitions $(n_i)_{i=0}^r$ of n where $n_0 \geqslant s$ can fully describe the structure of many MNH graphs G. Namely, each such G contains $r+1$ mutually disjoint complete graphs $K_{n_i}^{(i)}$, $i = 0, 1, \ldots, r$, with s vertices of $K_{n_0}^{(0)}$, labelled with $v_1, v_2, \ldots, v_s$. Moreover, each of additional edges of G is incident to some v_k according to the rule that all vertices of $K_{n_i}^{(i)}$, $i \geqslant 1$, are adjacent in G to v_j if and only if a_{ij} in A is 1. Note that for each $A = A_\alpha$ above each acceptable partition of n determines a 1-tough MNH graph, which we shall call an A-*graph*. Notice also that the simplest 1-tough nontrivially MNH graphs (which has $n = 7$

vertices and was found by Chvátal [2]) is an A-graph whose structure is described by A_1 ($\alpha = 1, r = 3, s = 4$) and the sequence $(4, 1, 1, 1)$, the unique acceptable ordered partition of 7.

The second subclass consists of WM-graphs which belong to the class of graphs studied by Watkins and Mesner [8]. These are unions of five complete graphs $K_{n_i}^{(i)}$ ($i = 1, 2, \ldots, 5$) such that the order $n_i \geq 3$ for each i, the first two graphs as well as the three remaining ones are mutually disjoint, and, for each $i \leq 2$ and each $j \geq 3$, $K_{n_i}^{(i)}$ shares precisely one vertex with $K_{n_j}^{(j)}$. There is a 1-1 correspondence between WM-graphs on n vertices and the collection of all pairs of sets $(\{n_1, n_2\}, \{n_3, n_4, n_5\})$ with $n = (\sum_i n_i) - 6$, $n_i \geq 3$ $(n \geq 9)$.

The simplest of WM-graphs (on $n = 9$ vertices) is described by the pair $(\{3\}, \{3\})$ with $n = 9$, or the sequence $(3, 3, 3, 3, 3)$. It is the unique MNH homogeneously traceable graph on 9 vertices, found independently by Skupień (see [5]).

All remaining 1-tough MNH graphs (including K_1 and K_2) are called R-graphs. For $3 \leq n \leq 10$, there are three such graphs (all of order $n = 10$): the notorious Petersen graph and two graphs depicted in Fig. 1 and 2. Note that the graph in Fig. 2 was found independently by Skupień [7] as the smallest MNH homogeneously traceable graph G with $\Delta(G) = n - 4$ (as well as with $\Delta(G) + \delta(G) = n - 2$).

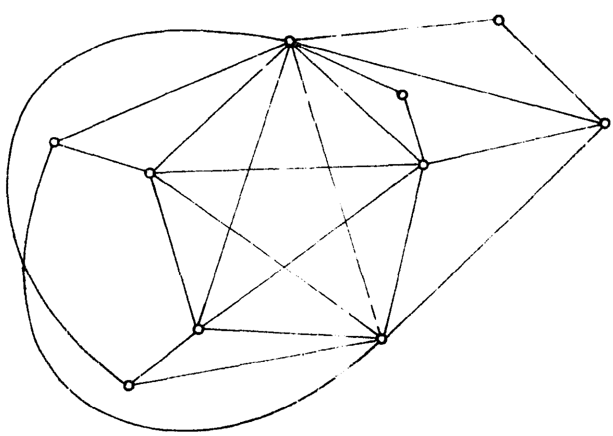

Fig. 1.

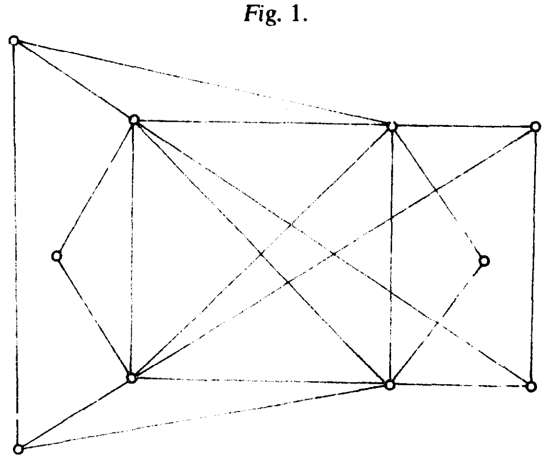

Fig. 2.

Table 2. Numbers of MNH graphs.

n	1	2	3	4	5	6	7	8	9	10
i_n	-	-	1	1	3	3	6	7	11	13
a_n	-	-	-	-	-	-	1	2	6	13
w_n	-	-	-	-	-	-	-	-	1	2
r_n	1	1	-	-	-	-	-	-	-	3
m_n	1	1	1	1	3	3	7	9	18	31

Table 3. The list of MNH graphs G with $n \leqslant 10$.

	$s(G)=1$		A-graphs			WM-graphs		
n	κ	$n_1, n_2, \ldots, n_{\kappa+1}$	α	n_0	$n_1, \ldots, n_r$	n_1, n_2	n_3, n_4, n_5	R-graphs
1								K_1
2								K_2
3	1	1, 1						
4	1	2, 1						
5	1	3, 1; 2, 2						
	2	1, 1, 1						
6	1	4, 1; 3, 2						
	2	2, 1, 1						
7	1	5, 1; 4, 2; 3, 3	1	4	1, 1, 1			
	2	3, 1, 1; 2, 2, 1						
	3	1, 1, 1, 1						
8	1	6, 1; 5, 2; 4, 3	1	5	1, 1, 1			
	2	4, 1, 1; 3, 2, 1; 2, 2, 2		4	2, 1, 1			
	3	2, 1, 1, 1						
9	1	7, 1; 6, 2; 5, 3; 4, 4	1	6	1, 1, 1	3, 3	3, 3, 3	
	2	5, 1, 1; 4, 2, 1		5	2, 1, 1			
		3, 3, 1; 3, 2, 2		4	3, 1, 1; 2, 2, 1			
	3	3, 1, 1, 1; 2, 2, 1, 1	2	5	1, 1, 1, 1			
	4	1, 1, 1, 1, 1	3	5	1, 1, 1, 1			
10	1	8, 1; 7, 2; 6, 3; 5, 4	1	7	1, 1, 1	4, 3	3, 3, 3	(1) Fig. 1
	2	6, 1, 1; 5, 2, 1; 4, 3, 1		6	2, 1, 1	3, 3	4, 3, 3	(2) Fig. 2
		4, 2, 2; 3, 3, 2		5	3, 1, 1; 2, 2, 1			(3) Petersen graph
	3	4, 1, 1, 1; 3, 2, 1, 1		4	4, 1, 1; 3, 2, 1			
		2, 2, 2, 1			2, 2, 2			
	4	2, 1, 1, 1, 1	2	6	1, 1, 1, 1			
				5	2, 1, 1, 1; 1, 1, 1, 2			
			3	6	1, 1, 1, 1			
				5	2, 1, 1, 1; 1, 1, 1, 2			

In Table 2, $m_n = i_n + a_n + w_n + r_n$ is the sum of numbers of MNH graphs of order n which have scattering number 1 (i_n), are A-graphs (a_n), WM-graphs (w_n), or R-graphs (r_n), respectively.

Table 3 gives a list of all MNH graphs G with $n \leqslant 10$.

Acknowledgement

Thanks are due to one of the referees who suggested making use of matrices A_α and drew our attention to Watkins and Mesner's results.

References

[1] J.A. Bondy and V. Chvátal, A method in graph theory, Discrete Math. 15 (1976) 111–135.
[2] V. Chvátal, Tough graphs and hamiltonian circuits, Discrete Math. 5 (1973) 215–228.
[3] A.M. Hobbs, A catalog of minimal blocks, J. Res. Nat. Bur. Stands. 77B (1973) 53–60.
[4] H.A. Jung, On a class of posets and the corresponding comparability graphs, J. Comb. Theory (B) 24 (1978) 125–133.
[5] Z. Skupień, On maximal nonhamiltonian graphs, Rostock. Math. Kolloq. 11 (1979) 97–106.
[6] Z. Skupień, Degrees in homogeneously traceable graphs, Proc. Conf. Montreal, 1979, in: Annals Discrete Math. 8 (1980) 185–188.
[7] Z. Skupień, Maximum degree among vertices of a non-Hamiltonian homogeneously traceable graph, to appear.
[8] M.E. Watkins and D.M. Mesner, Cycles and connectivity in graphs, Canad. J. Math. 19 (1967) 1319–1328.